Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A Property of Lines in n-Dimensional Space.

By E. O. Lovett, *Princeton, New Jersey.*

---

1. The following geometrical facts are well known. A line, i. e., a one-dimensional manifoldness, in the plane has curvature when not every three consecutive points on it are collinear; a line in ordinary space has double curvature when not every four points of it are coplanar; similarly, a line in a flat space of four dimensions is said to have triple curvature when not every five consecutive points of it lie in a Euclidean space of four dimensions; and so on for spaces of any number of dimensions.

If a secant line MM' approach a limiting position as M' approaches M on the curve, the limiting position is called a tangent to the curve at the point M; if a secant plane $MM'M''$ tend to assume a limiting position as M' and M'' approach M, the limiting plane is the osculating plane of the curve at the point M; finally, if the Euclidean space Σ_3, $MM'M''M'''$ tend to a limiting position as $M'M''$ and M''' approach coincidence with M, the limiting Σ_3 is called the osculating Σ_3 of the curve at the point M.

Two consecutive tangents have one point in common; two consecutive osculating planes intersect in a tangent, and three consecutive osculating planes have a common point; two, three or four consecutive osculating Σ_3's have in common, respectively, an osculating plane, a tangent, and a point, of the curve.

Every right line perpendicular to the tangent at M at the point of tangency is by definition a normal to the curve at M; a curve of triple curvature possesses a doubly infinite number of normals at every point; these all lie in a Σ_3 which is called the normal space. Of these ∞^2 normals one only is found in the osculating plane; it is called the principal normal. A simply infinite number of normals are perpendicular to the osculating plane; these are called binormals because each one is perpendicular to two consecutive tangents. One binormal, the principal binormal, is contained in the osculating space, and one, the trinormal, is perpendicular to the osculating space; the latter is so called because it is normal to three consecutive tangents.

The tangent, principal normal, principal binormal, and trinormal at the same point are the principal directions of the curve at this point; they are mutually perpendicular, and each is normal to the Σ_3 formed by the other three. The curve admits of but one osculating plane at a point, but has a simply infinite number of normal planes, all of which are situated in the normal space; one of the normal planes, the principal normal plane, is in the osculating space. The principal normal plane is determined by the principal normal and the principal binormal, since the osculating and normal spaces have the principal normal and principal binormal in common. The normal space is the space perpendicular to the tangent; the space perpendicular to the trinormal is the osculating space; that normal to the principal normal is called the rectifying space; there is no occasion for giving the fourth space a name.

Thus, at every point of the curve there is a tetrarectangular tetraeder whose edges are the four remarkable lines associated with the curve at that point, namely, the tangent, the principal normal, the principal binormal, and the trinormal. It is convenient to assume as origin the movable point M of the curve, the tangent as x-axis, the trinormal as y-axis, the principal binormal as z-axis, and the principal normal as t-axis. Let $\delta\phi$ be the angle between the two consecutive tangents, $\delta\psi$ that between the two adjacent principal binormals, and $\delta\chi$ that between the two adjacent trinormals, when the origin M is shifted along the curve to the neighboring position M'. Then, from the definitions of the principal axes,

$$\cos(t, x') = d\phi, \quad \cos(t, z') = d\psi, \quad \cos(z, y') = d\chi,$$

In order to complete the array of direction-cosines of the axes at M' with respect to those at M, it is only necessary to observe that

$$\cos(x, x') = \cos(y, y') = \cos(z, z') = \cos(t, t') = 1,$$

and that because of the perpendicularity of the axes

$$\cos(x, y') = 0, \quad \cos(x, z') = 0, \quad \cos(x, t') = -d\phi, \; \ldots \; ;$$

accordingly, the quadrangle of direction-cosines becomes

$$\left.\begin{array}{c|cccc} & x & y & z & t \\ \hline x' & 1 & 0 & 0 & d\phi \\ y' & 0 & 1 & d\chi & 0 \\ z' & 0 & -d\chi & 1 & d\psi \\ t' & -d\phi & 0 & -d\psi & 1 \end{array}\right\} \tag{1}$$

The ratios of the differentials of ϕ, ψ, χ to ds, i. e., the limits of the ratios of $\delta\phi$, $\delta\psi$, $\delta\chi$ to MM' as M' approaches M, measure the curvatures at M; the first is called the flexion, the second the torsion, and the third the curvature; or perhaps more simply, the first, second and third curvatures. Thus, if

$$ds = \rho d\phi = \tau d\psi = r d\chi,$$

the number ρ, τ, r, the reciprocals of the curvatures, measure three lengths which are called, respectively, the radius of flexion, the radius of torsion, and the radius of curvature. The flexion of a space curve consists of the more or less rapid deviation of the curve from the tangent; the torsion, of the more or less swing from the osculating plane; the curvature, of the rapidity with which the curve leaves the osculating space.

The array (1) shows that in order to discuss the curve in the domain of each of its points, it is sufficient to know the functions ϕ, ψ, χ, i. e., sufficient to give ρ, τ, r as functions of s. Thus, the three equations, the so-called intrinsic equations of the curve

$$f_1(s, \rho, \tau, r) = 0, \quad f_2(s, \rho, \tau, r) = 0, \quad f_3(s, \rho, \tau, r) = 0,$$

determine the curvatures at every point of the curve lying between points at which the tangents may become indeterminate; the equations determine the forms of these arcs but do not fix the positions of the branches in space.

2. The method of the intrinsic analysis used so successfully by Professor E. Cesàro in his "Lezioni di geometria intrinseca" and developed by him for configurations in spaces of any number of dimensions, may be employed to bring to light an interesting property of curves in n-dimensional space.

Let P be a point in space referred to M as origin and generally movable with M; let its coordinates be $x(s)$, $y(s)$, $z(s)$, $t(s)$. Let P' be the point of the trajectory of P corresponding to M'; its coordinates referred to axes at M are $x + \delta x$, $y + \delta y$, $z + \delta z$, $t + \delta t$; its coordinates referred to axes at M' are $x + dx$, $y + dy$, $z + dz$, $t + dt$. Let u, v, w, p be the coordinates of M' referred to M. Then projecting $M'P'$ on the axes of M, we have, by virtue of the array (1),

$$\left.\begin{aligned} x + \delta x &= u + x + dx - (t + dt)\, d\phi, \\ y + \delta y &= v + y + dy - (z + dz)\, d\chi, \\ z + \delta z &= w + z + dz + (y + dy)\, d\chi - (t + dt)\, d\psi, \\ t + \delta t &= p + t + dt + (x + dx)\, d\phi + (z + dz)\, d\psi. \end{aligned}\right\} \tag{2}$$

Confining attention to those curves, for which it is legitimate to assume that the limit of the ratio of the arc to the chord is unity, it follows from the definition of the tangent that

$$\mathcal{L}\frac{u}{k} = 1, \quad \mathcal{L}\frac{v}{k} = \mathcal{L}\frac{w}{k} = \mathcal{L}\frac{p}{k} = 0, \quad k = \sqrt{u^2 + v^2 + w^2 + p^2},$$

hence

$$\mathcal{L}\frac{u}{\delta s} = 1, \quad \mathcal{L}\frac{v}{\delta s} = \mathcal{L}\frac{w}{\delta s} = \mathcal{L}\frac{p}{\delta s} = 0,$$

and the formulæ (2) give the fundamental relations of Cesàro's analysis,

$$\left.\begin{aligned} \frac{\delta x}{ds} &= \frac{dx}{ds} - \frac{t}{\rho} + 1, & \frac{\delta y}{ds} &= \frac{dy}{ds} - \frac{z}{r}, \\ \frac{\delta z}{ds} &= \frac{dz}{ds} + \frac{y}{r} - \frac{t}{\tau}, & \frac{\delta t}{ds} &= \frac{dt}{ds} + \frac{x}{\rho} + \frac{z}{\tau}. \end{aligned}\right\} \tag{3}$$

The latter formulæ give the following as the necessary and sufficient conditions for the immobility of a point, viz.:

$$\frac{dx}{ds} = \frac{t}{\rho} - 1, \quad \frac{dy}{ds} = \frac{z}{r}, \quad \frac{dz}{ds} = \frac{t}{\tau} - \frac{y}{r}, \quad \frac{dt}{ds} = -\frac{x}{\rho} - \frac{z}{\tau}. \tag{4}$$

Now, take the origin of arcs at any point of the curve at which the curvatures, always supposed to be continuous functions of the arc, have finite values; let x, y, z, t be its coordinates with respect to a fundamental tetraeder at a neighboring point. The coordinates x, y, z, t are infinitesimal along with s, and since the conditions (4) ought to be satisfied, we have

$$\mathcal{L}\frac{x}{s} = \mathcal{L}\frac{dx}{ds} = \mathcal{L}\left(\frac{t}{\rho} - 1\right) = -1, \quad \mathcal{L}\frac{y}{s} = \mathcal{L}\frac{dy}{ds} = \mathcal{L}\frac{z}{r} = 0,$$

$$\mathcal{L}\frac{z}{s} = \mathcal{L}\frac{dz}{ds} = \mathcal{L}\left(\frac{t}{\tau} - \frac{y}{r}\right) = 0, \quad \mathcal{L}\frac{t}{s} = \mathcal{L}\frac{dt}{ds} = \mathcal{L}\left(\frac{x}{\rho} + \frac{z}{\tau}\right) = 0,$$

then, also,

$$\mathcal{L}\frac{t}{s^2}=\tfrac{1}{2}\mathcal{L}\frac{1}{s}\,\frac{dt}{ds}=-\tfrac{1}{2}\mathcal{L}\frac{1}{s}\left(\frac{x}{\rho}+\frac{z}{\tau}\right)=\frac{1}{2\rho},$$

$$\mathcal{L}\frac{z}{s^3}=\tfrac{1}{3}\mathcal{L}\frac{1}{s^2}\left(\frac{t}{\tau}-\frac{y}{r}\right)=\frac{1}{6\rho\tau},$$

$$\mathcal{L}\frac{y}{s^4}=\tfrac{1}{4}\mathcal{L}\frac{1}{s^3}\left(\frac{z}{r}\right)=\frac{1}{24\rho\tau r}.$$

The expressions for the coordinates u, v, w, p of M', with regard to the axes of M, are obtained by changing s into $-ds$ in the above formulæ, since transferring the origin of arcs to the point M' $(s+ds)$ infinitely near to $M(s)$ is equivalent to putting $s+ds=0$; accordingly,

$$u=ds,\quad v=\frac{ds^4}{24\rho\tau r},\quad w=-\frac{ds^3}{6\rho\tau},\quad p=\frac{ds^2}{\rho}; \tag{5}$$

hence, of all the Σ_3's passing through M, those containing the tangent are characterized by the fact that their distances from points on the curve infinitely near to M are infinitesimals of a higher order, and for one of these, the osculating Σ_3, the distance is infinitesimal of at least the fourth order.

It follows also that every arc, sufficiently small, taken in the neighborhood of the point M is situated wholly on one side of any Σ_3 through the tangent, with the exception of the Σ_3 formed by the tangent, trinormal, and principal binormal. Contrary, then, to the property of curves of double curvature, the osculating space of highest dimensions of a curve of triple curvature lies wholly on one side of the curve. The process by which this result was reached shows that a line of multiple curvature cuts its osculating space of highest dimensions or lies wholly on one side of that space according as the number of dimensions of the space necessary to the existence of the curve is odd or even.